UDC 551.513.1:551.515.5:551.557.1

THE INFLUENCE OF LATITUDINAL WIND SHEAR UPON LARGE-SCALE WAVE PROPAGATION INTO THE TROPICS

JOHN R. BENNETT and JOHN A. YOUNG

The University of Wisconsin, Madison, Wis.

ABSTRACT

The effects of horizontal shear of the mean zonal wind on the lateral propagation of disturbances through the Tropics is studied by the use of a one-layer model. The governing equations are reduced to a second-order differential equation for v, the northward component of velocity. The equation is analyzed as an eigenvalue problem and solved numerically for the free modes of the Tropics for the case with zero mean flow. These solutions are compared with solutions that are forced at a boundary situated in mid-latitudes, for cases with and without a mean zonal flow.

At "critical latitudes," the basic equation has a singularity (where the phase speed of a wave forced at the boundary is equal to the mean flow). The case for forced motions is investigated in more detail by numerically studying the evolution of disturbances as an initial value problem for the case of nondivergent flow.

The horizontal shear is shown to significantly alter the types of mid-latitude motions that can affect tropical motions. In particular, disturbances with large eastward phase propagation are shown to have negligible effect. Disturbances that have phase speeds that are somewhere equal to the mean flow are shown to be absorbed at the critical latitude. Disturbances with phase speeds more westward than the mean flow may be free to propagate into the Tropics, providing their wavelengths are not too short.

1. INTRODUCTION

In the last few years, there has been renewed interest in the study of large-scale tropical wave motions. To a certain extent, the low spatial resolution of the observational network in the Tropics has been offset by the use of analysis of long time series of wind data at a few stations (Rosenthal 1960, Maruyama 1968). Theoretical works by Rosenthal (1965), Matsuno (1966), Koss (1967), and Lindzen (1967) have been of great value in interpreting the data, since tropical motions may consist of several types of wave disturbances (such as Kelvin waves and mixed Rossby-gravity waves) that do not occur elsewhere. In a recent summary, Wallace (1969) described the results of observations of the Kelvin-type waves, the mixed Rossby-gravity modes (which are vertically propagating and occur mainly in the stratosphere), and two types of tropospheric modes that do not propagate vertically. With the exception of one of the tropospheric modes that has a wavelength of 3000 km, the other waves have wavelengths in excess of 10,000 km.

The sources of energy for synoptic scale Rossby wave motions at low latitudes is poorly understood at present. Suggested mechanisms have included tropical instabilities associated with latent heating or horizontal wind shear and the lateral coupling of the Tropics with mid-latitude waves. For example, Charney and Eliassen (1964) and Rosenthal (1967) first studied the stability of quasi-balanced disturbances driven by organized convection; research on this conditional instability of the second kind is presently being continued by several investigators. Investigations into the possibility of barotropic instability at low latitudes have recently been accomplished by Nitta and Yanai (1969) and Lipps (1970).

The lateral coupling mechanism was proposed by Mak (1969) and demonstrated to be plausible in a simple two-layer stochastic model. A prominent dynamical influence in the model was the horizontal shear of the zonal wind. However, Mak chose to emphasize the statistics of the motions and did not attempt to explain the results in physical detail. This paper is an attempt to isolate the various effects of horizontal shear on this process of energy exchange between the Tropics and mid-latitudes.

The propagation of energy through a shear flow arises in other physical problems such as the generation of surface water waves (Miles 1962) and the upward propagation of gravity waves in the atmosphere. Internal gravity wave theory (Bretherton 1966) predicts that certain layers (where the mean wind is equal to the trace speed of the wave) tend to absorb wave energy. Dickinson (1970) has recently studied Rossby wave absorption in the immediate vicinity of such a region. Other layers with varying mean flow profiles also redistribute the energy through the processes of reflection and tunneling.

Charney (1969) discussed the problem of energy propagation into the Tropics qualitatively by the use of WKB methods (Morse and Feshbach 1953). His main conclusion was that westerly disturbances should not propagate far into an easterly regime. Mak's results are consistent with this idea in that the forced motions in the Tropics of his model were found to exhibit predominantly easterly phase propagation, even when the bulk of the mid-latitude wave forcing was westerly.

The above work indicates that the characteristics of the mean flow may highly restrict the character of that lateral forcing which is likely to produce effects in the Tropics. The purpose of this work is to shed additional light on this interaction. The basic model is formulated, and a wave equation is derived from it in section 2. Section 3 introduces a method of analysis of the problem involving free and forced solutions and illustrates it for the case of zero mean zonal flow. Section 4 treats forced motion in the presence of a zonal flow for the cases in which there is no wave absorption, and comparison is made with the results of section 3. Section 5 contains a treatment of the case in which wave absorption occurs. The process of absorption is shown by solving for the establishment of a wave regime as an initial value problem.

2. FORMULATION OF THE MODEL

Since we intend to focus on the effects of horizontal shear on the lateral propagation of wave energy, we will consider a model that allows only quasi-horizontal motion. This limits the applicability of the results to motions occurring in deep atmospheric layers. But, because of the paucity of observations, only these largest modes have been observed in detail. Examples where this approximation is valid are the tropospheric modes that do not propagate vertically which were discussed by Wallace (1969). These wave motions can occur in a simple one-layer model on an equatorial beta-plane such as Matsuno (1966) used. We wish to alter this model systematically by adding the effects of a zonal shear flow.

We consider a single layer of a homogeneous, inviscid fluid with a free surface in hydrostatic equilibrium. The equations for the fluid are linearized about a steady base state consisting of zonal flow $U(y)$ that varies in the northward direction y. The linearized equations for the two horizontal velocity components (which depend on the two horizontal coordinates and time only) and the equation of continuity are

$$\frac{\partial u'}{\partial t}+U\frac{\partial u'}{\partial x}+v'\frac{dU}{dy}-fv'+\frac{\partial \phi'}{\partial x}=0, \tag{1}$$

$$\frac{\partial v'}{\partial t}+U\frac{\partial v'}{\partial x}+fu'+\frac{\partial \phi'}{\partial y}=0, \tag{2}$$

and

$$\frac{\partial \phi'}{\partial t}+U\frac{\partial \phi'}{\partial x}+v'\frac{\partial}{\partial y}(g'H)+g'H\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right)=0. \tag{3}$$

The primed quantities are perturbation variables, and the capital letters (U, H) refer to the base state. In eq (3), g' is a scaled acceleration due to gravity. In our application of the model to the earth's atmosphere, Matsuno (1966) has shown that it may be thought of as a value reduced as follows:

$$g'=g\left(\frac{\Delta\bar{\theta}}{\bar{\theta}}\right)$$

where $\bar{\theta}$ is the mean potential temperature of the column and $\Delta\bar{\theta}$ is its difference over the vertical distance H.

Velocities in the x and y directions are u' and v', and the geopotential of the top surface is ϕ $(=g'z)$.

Because we are only considering deep motions, H can be replaced by H_0, an average of H over the whole domain of interest, when it is multiplied by the divergence in eq (3). The Coriolis parameter, f, is assumed to vary linearly in the northward direction:

$$f=\beta y \tag{4}$$

where $\beta=df/dy=2.29\times 10^{-11}\ \mathrm{m}^{-1}\,\mathrm{s}^{-1}$ at the Equator. From the assumptions that the base state is composed of a steady zonal velocity that varies only with y, it follows that it is geostrophic:

$$\frac{dH}{dy}=-\frac{f}{g'}U. \tag{5}$$

Equations (1), (2), and (3) can be made nondimensional by choosing time and length scales of

$$\hat{T}=(g'H_0)^{-1/4}\beta^{-1/2},\ \hat{L}=(g'H_0)^{1/4}\beta^{-1/2}. \tag{6}$$

It follows that the velocity scale is $\hat{L}/\hat{T}=(g'H_0)^{1/2}$, the speed of long gravity waves in the model when β is set equal to zero. Using eq (4), (5), and (6), the equations for the nondimensional variables take the form

$$\frac{\partial u'}{\partial t}+U\frac{\partial u'}{\partial x}+v'\frac{dU}{dy}-yv'+\frac{\partial \phi'}{\partial x}=0, \tag{7}$$

$$\frac{\partial v'}{\partial t}+U\frac{\partial v'}{\partial x}+yu'+\frac{\partial \phi'}{\partial y}=0, \tag{8}$$

and

$$\frac{\partial \phi'}{\partial t}+U\frac{\partial \phi'}{\partial x}-yUv'+\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}=0. \tag{9}$$

Assume solutions of the form

$$u'=\mathrm{Re}[u(y)e^{i(kx-kct)}],$$

$$v'=\mathrm{Re}[v(y)e^{i(kx-kct)}],$$

and

$$\phi'=\mathrm{Re}[\phi(y)e^{i(kx-kct)}].$$

The equations then reduce to

$$ik(U-c)u+v\frac{dU}{dy}-yv+ik\phi=0, \tag{10}$$

$$ik(U-c)v+yu+\frac{d\phi}{dy}=0, \tag{11}$$

and

$$ik(U-c)\phi-yUv+iku+\frac{dv}{dy}=0. \tag{12}$$

By solving eq (10) for U and substituting in the last two equations, we obtain

$$ik(U-c)v+\frac{y}{ik(U-c)}\left[yv-ik\phi-v\frac{dU}{dy}\right]+\frac{d\phi}{dy}=0 \tag{13}$$

and

$$ik(U-c)\phi-yUv+\frac{1}{U-c}\left[yv-ik\phi-v\frac{dU}{dy}\right]+\frac{dv}{dy}=0. \tag{14}$$

Differentiation of eq (14) with respect to y and substitution for $d\phi/dy$ from eq (13) and for ϕ from the undifferentiated form of eq (14) yields a single equation for v:

$$\frac{d^2v}{dy^2}+\frac{dv}{dy}\left[-yU-\frac{2(U-c)}{(U-c)^2-1}\frac{dU}{dy}\right]$$
$$+v\left[k^2(U-c)^2-k^2+\frac{1}{U-c}-y^2-U-\frac{1}{U-c}\frac{d^2U}{dy^2}\right.$$
$$+\left(\frac{dU}{dy}\right)^2\left(\frac{1}{(U-c)^2}\right)\left(1-\frac{1}{1-(U-c)^2}\right)+y^2U/(U-c)$$
$$+\frac{1}{(U-c)^2-1}\frac{dU}{dy}\left\langle yU(U-c)+yU/(U-c)\right.$$
$$\left.\left.+\frac{dU}{dy}-2y\right\rangle\right]=0. \quad (15)$$

Equation (15) is the general governing equation for waves in the Tropics for this model. Because the speeds of different wave types vary so greatly, the terms in eq (15) are not of uniform significance. For example, the approximation $c\gg U$ may be a good one for gravity waves. On the other hand, our interest in the slower moving waves (quasi-geostrophic away from the Equator) leads us to explore the case $c\sim U$. Thus under the conditions

$$c=O(U),\ c\ll 1,\ \frac{dU}{dy}=O\left(\frac{U}{L}\right),\ y=O(L)$$

and using the fact that $[1/(U-c)^2]\{1-1/[1-(U-c)^2]\}$ has magnitude unity, one can scale the terms of eq (15).

$$\begin{array}{ccccccc}
\underline{1} & L^2U & U^2 & & & & \\
 & k^2L^2U^2 & \underline{k^2L^2} & L^2/\underline{(U-c)} & \underline{L^4} & UL^2 & U/\underline{(U-c)} \\
 & & U^2 & & & \underline{UL^4/(U-c)} & \\
 & & & U^3L^2 & & L^2U^2/(U-c) & \\
 & & & & & U^2 & UL^2
\end{array}$$

The U and L are characteristic nondimensional velocities and lengths. For flows in which $\sqrt{g'H_0}=100$ m s^{-1} ($\Delta\bar{\theta}/\bar{\theta}=0.1$, $H_0=10$ km), the length scale from eq (6) is approximately 2100 km. If we consider this length scale as unity and assume $U\approx 0.1$ and $k\approx 1.0$, the dominant terms in eq (15) (with scalings underlined) give

$$\frac{d^2v}{dy^2}+v\left(-k^2-y^2+\frac{1}{(U-c)}\left[1-\frac{d^2U}{dy^2}+y^2U\right]\right)=0. \quad (16)$$

This equation is the approximate governing equation for the propagation of large-scale, slowly moving wave motions in the Tropics.

This equation may be thought of as a generalization of the equation derived by Matsuno (for the case $U\equiv 0$) to cases involving mean zonal shear flows. We consider the kinds of solutions to eq (16) in the following two sections. Section 3 considers the case $U=0$, whereas section 4 considers the more general case $U(y)\neq 0$.

3. FREE AND FORCED WAVES IN THE ABSENCE OF A MEAN ZONAL FLOW

The analysis of the last section developed a second-order ordinary differential equation for v, the complex representation of the northward component of velocity of the form

$$\frac{d^2v}{dy^2}+Q(k,\ c,\ U,y)v=0.$$

We now wish to use this equation to investigate the response of the Tropics to forcing at the northern and southern boundaries. Since the equation is linear, we can consider the forcing composed of a superposition of different wave numbers, k, and phase speeds, c. The response of the Tropics will then be the sum of the responses to the various boundary modes. Moreover, it is possible to separate the effects of the two boundaries by setting $v=0$ at one boundary and $v\neq 0$ at the other and adding the response to the solutions obtained by reversing the procedure. This procedure of adding solutions obtained from the two boundaries for various frequencies ($=kc$) and wave numbers, k, has been used by Mak (1969) for a two-layer model with internal dissipation. Mak did not explain physically why the eastward-propagating waves at the boundary (which are typical of mid-latitude motions) produced a negligible response in the Tropics. To isolate the effects of different forcing situations and the effect of the wind shear on the response, we first treat the problem without a zonal wind. Since we expect the forced response to be related to the free modes of the Tropics, we begin by examining them.

When the mean velocity is set identically equal to zero, eq (15) reduces to

$$\frac{d^2v}{dy^2}+\left(k^2c^2-k^2-\frac{1}{c}-y^2\right)v=0. \quad (17)$$

As shown in the last section, the term k^2c^2v is smaller than the other terms when we consider the slowly moving large-scale waves. To facilitate comparison with Matsuno's results, we will include this term for the time being. The free waves are described by the eigenfunctions of this equation satisfying the homogeneous boundary conditions. In the infinite domain, the solutions are $e^{-y^2/2}$ multiplied by Hermite (Abramowitz and Segun 1965) polynomials, $H_n(y)$; the eigenvalues are

$$\alpha=k^2c^2-k^2-\frac{1}{c}=2n+1, \qquad n=0,1,2,\ \ldots,$$

where n is the order of the polynomial. For each value of n, there are three roots for c, the phase speed. Two roots ($c\sim\pm\sqrt{1+(2n+1/k^2)}$) correspond to gravity waves, and one ($c\sim(-1/[k^2+2n+1])$) corresponds to a Rossby wave. Note that the neglect of the k^2c^2 term in eq (17) has the effect of filtering out the gravity wave solutions, for the characteristic equation then reduces to a linear (not cubic) form in which the sole root is that of the Rossby wave:

$$-k^2-\frac{1}{c}=2n+1, \quad n=0,1,2, \ldots.$$

As Matsuno pointed out, the low-order modes (small n) are trapped near the Equator (for $n=0$, $v=e^{-y^2/2}$) and are relatively insensitive to far-distant boundaries. However, the higher order modes found by Matsuno would be affected by the finiteness of the real earth. Theoretical evidence regarding such boundary effects has been presented by Rosenthal (1965) and Koss (1967).

To facilitate comparison with later numerical results, we now study the numerical eigenvalue problem corresponding to the homogeneous boundary conditions $v=0$ at $y=\pm 10.0$ by finding the eigenvalues and the eigenfunctions of the finite-difference analog of the homogeneous eq (17). Since eq (17) is symmetric with respect to the Equator, the solutions are either odd or even. Thus, in determining the eigenfunctions, we need only to compute the solution for half of the domain. We consider two eigenvalue problems as defined by the boundary conditions:

$$v(0)=0,\ v(10)=0 \text{ (the odd case)}$$

and

$$\frac{dv}{dy}(0)=0,\ v(10)=0 \text{ (the even case)}.$$

The finite-difference scheme gives the following algebraic equation for the value of v at the ith (i.e., $y=i\Delta y$) grid point in terms of the values of v on either side of it:

$$\frac{v_{i+1}+v_{i-1}-2v_i}{(\Delta y)^2}=-Q_i v_i.$$

This can be written as

$$\frac{v_{i+1}+v_{i-1}-2v_i}{(\Delta y)^2}-y_i^2 v_i=-\alpha v_i \tag{18}$$

where

$$\alpha=-k^2+k^2c^2-\frac{1}{c}.$$

The choice $\Delta y=0.5$ corresponds to 19 internal grid points between $y=0$ and $y=10$. Once the boundary conditions at $y=0$ and $y=10$ are specified, there are 19 algebraic equations for the 19 values of v, written here in matrix form:

$$\mathbf{A}\mathbf{v}=-\alpha\mathbf{v}. \tag{19}$$

$\mathbf{A}$ is a 19×19 matrix of the known coefficients, and $\mathbf{v}$ is a column vector of length 19. For the even case, $v(0)$ was set equal to v_1, the first value of v north of the Equator, to approximate the zero derivative at the Equator. Equation (19) can be written in the form

$$(\mathbf{A}+\alpha\mathbf{I})\mathbf{v}=0 \tag{20}$$

where $\mathbf{I}$ is the unit matrix. Since this is a homogeneous system of linear algebraic equations, for nontrivial solutions for v, the determinant of the matrix $(\mathbf{A}+\alpha\mathbf{I})$ must vanish. The values of α for which this occurs are the eigenvalues of the matrix $[-\mathbf{A}]$.

The eigenvalues and associated eigenvectors of the system (20) were determined numerically. Figure 1 exhibits some eigenfunctions that are symmetric about the Equator. We see that each mode is centered near a latitude $y=\alpha^{1/2}$. The highest modes (large α) are found away from the Equator and are understandably more nearly quasi-geostrophic in character. In these cases, the lowest frequency then reduces to the classical mid-latitude Rossby wave frequency

$$\omega=\frac{-\beta k}{k^2+l^2}$$

where $l^2=\alpha$.

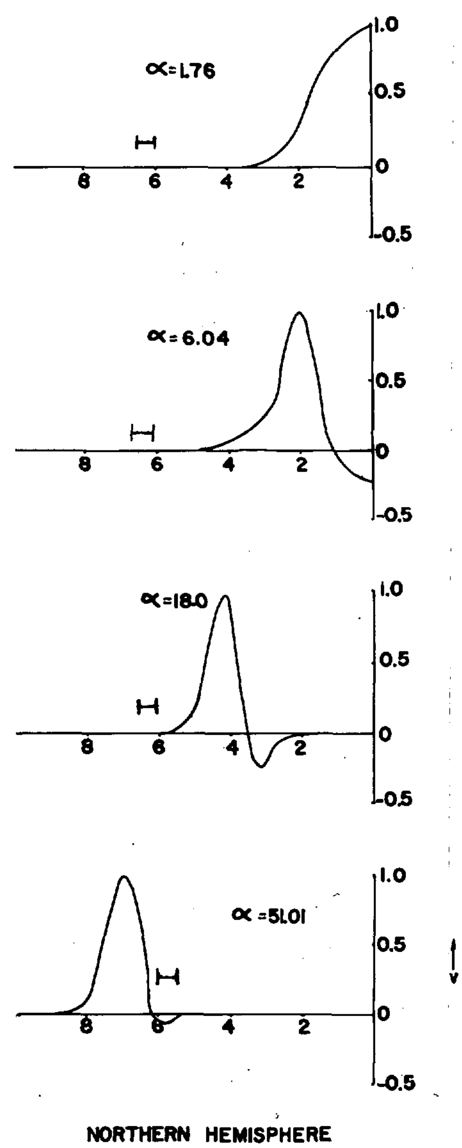

FIGURE 1.—Free modes of the Tropics, in the absence of a mean flow, that are symmetric about the Equator ($y=0$). The scale magnitude of v is arbitrary.

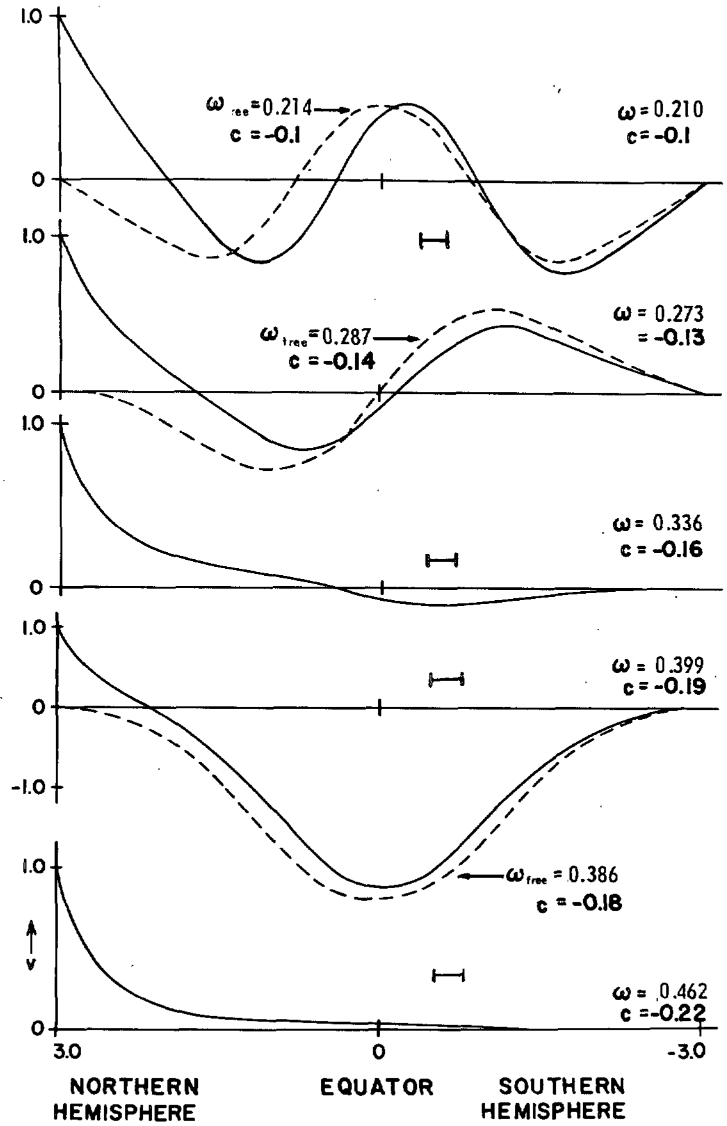

FIGURE 2.—Forced solutions for $v(y)$ for the case of zero mean flow. Where a resonant response occurs, the analogous free mode is given as a dashed line.

Additional numerical eigensolutions were found for the free waves for the case with boundaries at $y=\pm 3.0$. It was found that the trapped solutions with smallest α were not changed much by the closer boundaries. They are shown as dashed lines in figure 2 and will be discussed shortly. Solutions corresponding to larger values of α were so affected by truncation error as to have little value.

The effect of forcing in the Tropics in the absence of a zonal flow was investigated by specifying a nonzero value of v at the boundary $y=+3.0$ and setting $v=0$ at $y=-3.0$, thus modeling the effects upon the tropical atmosphere of a wave moving at higher latitudes in the Northern Hemisphere.

By arbitrarily setting the nonzero boundary condition and specifying c and k, we confine ourselves to looking at the asymptotic ($t\rightarrow\infty$) solution to the initial value problem generated by starting a wave traveling at constant speed and amplitude along the northern boundary until the whole tropical atmosphere is pulsing at its frequency and wavelength. By ignoring the transient responses to the forcing at the boundary, we are neglecting the way in which the energy initially enters the Tropics, preferring to look at the Tropics a long time later when a steady energy density has developed.

The forced solutions (with $U=0$) are shown as solid lines in figure 2 for various values of the phase speed. We see that the ability of the motions to penetrate into the Southern Hemisphere depends strongly on the phase speed (in the cases shown, all are to the west). The maximum response at all latitudes is found for waves in which phase speeds are close to those of a free mode, all of which move westward. In these quasi-resonant cases, we see that the structure of the response is very nearly the same as that of the free mode, as expected. Figure 2 also exhibits two "nonpropagating" responses ($c=0.16$, $c=0.22$) that do not resemble a wavelike structure in y and are of small amplitude throughout most of the Tropics.

4. FORCED WAVES IN THE PRESENCE OF A ZONAL CURRENT

The analysis in section 3 using eq (16) cannot completely be extended to the case in which there is a zonal current, because it is not possible to set up a conventional matrix eigenvalue problem for the free modes using eq (16) alone. It is possible to use the complete set of eq (10–12) as a basis for an eigenvalue approach, but the problem then becomes three times larger.

It is of course possible to use eq (16) to treat the forced problem, in which c is specified a priori. The problem can be analyzed qualitatively in the following manner. Equation (16) can be written as

$$\frac{d^2v}{dy^2}+Q(k,U,c,y)\,v=0. \tag{21}$$

The solutions can be broadly classified as being of three types: (1) propagating, (2) evanescent, and (3) singular. In regions where Q is positive, there are oscillatory solutions and thus latitudinal propagation. In regions where Q is negative, the wave is evanescent, since solutions forced at one boundary die off exponentially in y (recall the case $c=-0.22$ in fig. 2). In regions where $[U(y)-c]$ approaches zero, the coefficient Q gets arbitrarily large since it contains $1/(U-c)$. The equation has a regular singular point there. Since this case is difficult to treat numerically by the method we have employed so far, it will be the subject of the next section. In this section, we will try to learn something of the nonsingular solutions.

We will discuss a special case in which a parabolic zonal flow and a wave number k are specified. A fair approximation of the mean winds throughout the troposphere (Mak 1969) is

$$U(y)=-0.04+0.0625y^2 \tag{22}$$

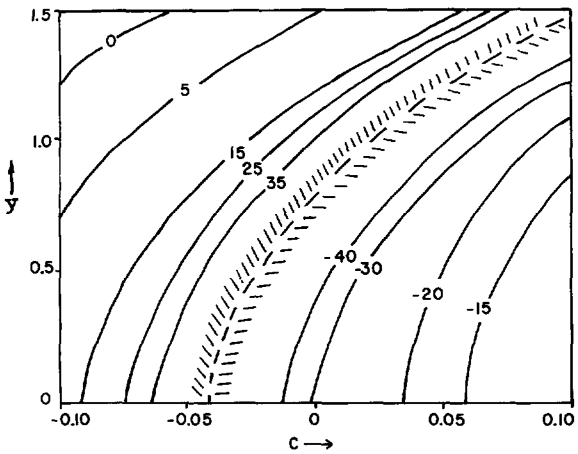

FIGURE 3.—Values of $Q(y, k, U, c)$ for $k=2.1$ and the mean flow given as a dashed line; $y=1.5$ corresponds to approximately 30° of latitude; positive values indicate propagating solutions while negative values indicate evanescent solutions. In the hatched region, $Q \sim 1/(U-c)$ and thus becomes arbitrarily large.

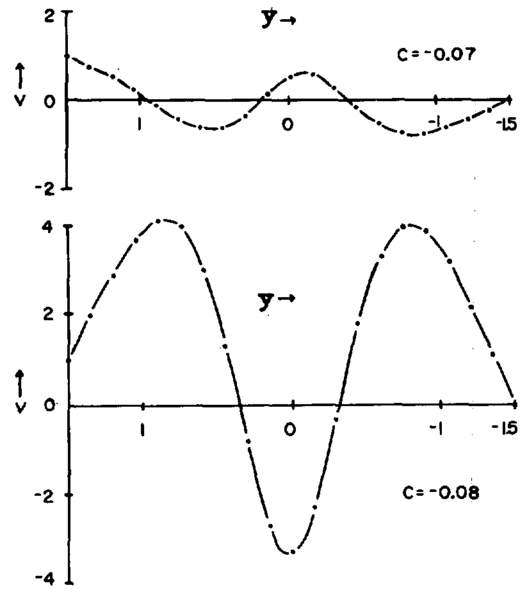

FIGURE 5.—Same as figure 4, except for different values of c.

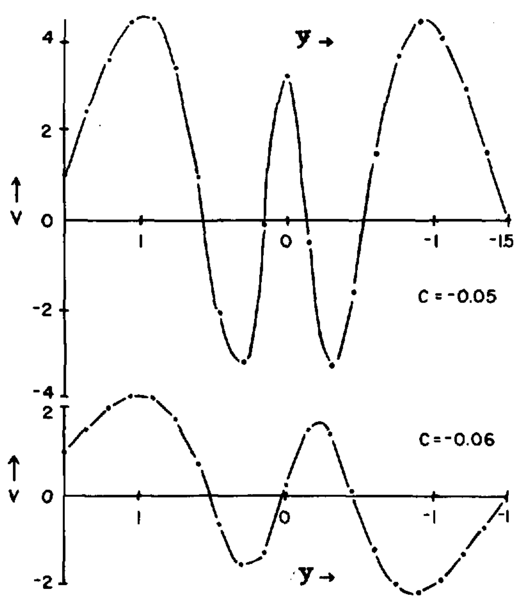

FIGURE 4.—Solutions of $v(y)$ where v is forced to be 1.0 at the northern boundary, for different values of c. The mean flow $U(y)$ and k $(=2.1)$ correspond to values in figure 3.

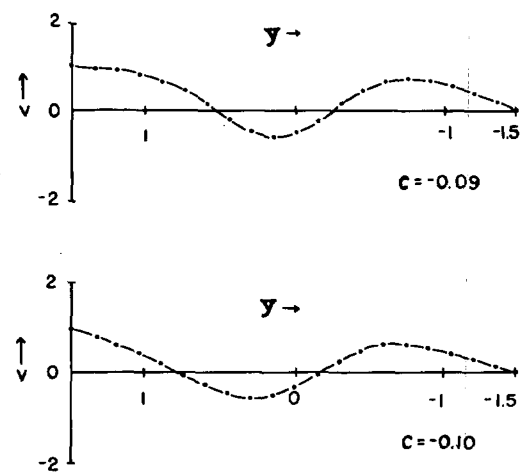

FIGURE 6.—Same as figure 4, except for different values of c.

where we have again chosen 100 m s^{-1} for $\sqrt{g'H_0}$. This gives a 4 m s^{-1} easterly wind at the Equator and a 10 m s^{-1} westerly wind at $y=1.5$ ($\sim$ 30°N).

Figure 3 shows values of Q as a function of c and y when $k=2.1$ (corresponding to an east-west wavelength of 6000 km) for the zonal flow specified in eq (22). The zonal flow is also plotted on the diagram as a dashed line. Note that the range of c for which $(U(y)-c)=0$ at some latitude is hatched, indicating that the equation becomes singular at the dashed line. Regions where Q is positive for the entire range of y occur only for waves moving westward faster than the maximum easterly zonal current, but more slowly than about 7 m s^{-1}.

Waves moving westward faster than 7 m s^{-1} must first "tunnel" their way through the evanescent region near the northern boundary. Inspection of eq (16) shows that the evanescent region expands downward and in the direction of increasing c as the x wavelength decreases (as k increases). Thus, it appears that only the longer waves can propagate freely into the Tropics; sufficiently short waves may suffer attenuation for all wave speeds.

Forced solutions were computed for nine different values of c in the propagating region for this current. They are shown in figures 4 through 8.

We see that, in many propagation cases, the wavelength changes significantly with latitude, implying a

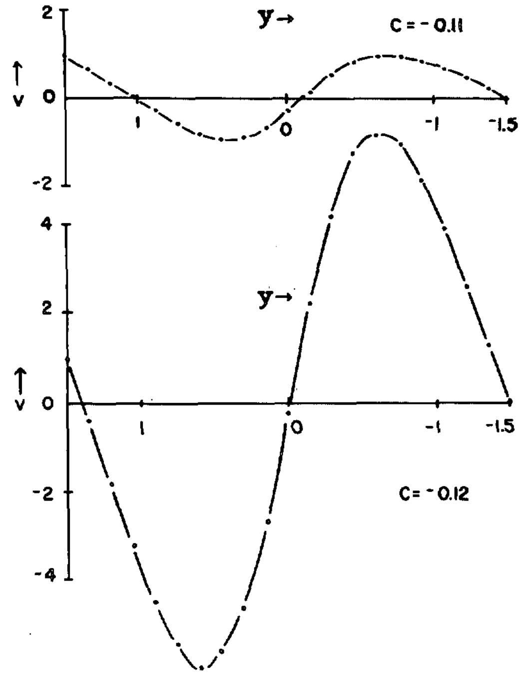

FIGURE 7.—Same as figure 4, except for different values of c.

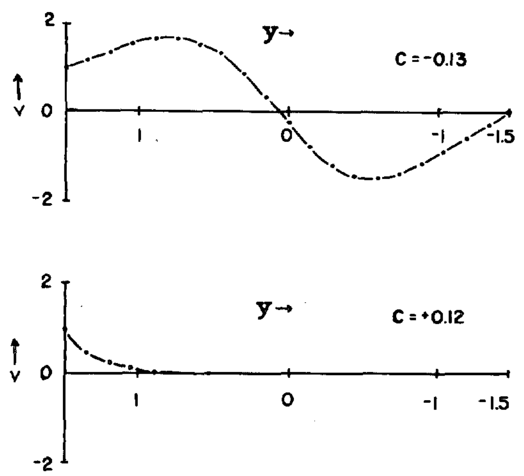

FIGURE 8.—Same as figure 4, except for different values of c.

continuous process of reflection and transmission that also alters the amplitude variations. The evanescent region near the left boundary is too weak to detect in most of the cases shown in figures 6–8. The gross magnitude of the response follows the same general trend as if the free modes were those in the problem treated in section 3. There are four responses for which a value of v is greater than that imposed on the boundary (figs. 4, 5, 7, and 8) and can therefore be called resonant. The ones with the most detailed structure (corresponding to the higher order polynomials of Matsuno) have the lower phase speeds. Under the reasonable assumption that figures 4, 5, 7, and 8 correspond to $n=4$, 3, 2, and 1, respectively, it is possible to compute the theoretical phase speed using Matsuno's approximate formula

TABLE 1.—*Comparison of low-frequency wave speeds taken from the advective model with Matsuno's approximate expression for the case without zonal current*

Description	n→	1	2	3	4
Matsuno wave with no zonal current		−0.14	−0.10	−0.08	−0.07
Model with zonal current		−.12	−.08	−.06	−.05

$$c=-\frac{1}{k^2+2n+1} \tag{23}$$

for free Rossby modes and compare it with the speeds for our results (table 1).

Note that the free modes are shifted in phase velocity in the direction of the average of the wind over the Tropics ($U(y)$ averaged from $y=0.0$ to $y=1.5$ is $+0.034$, i.e., westerly). This implies that the net effect of the shear flow is to cause a larger difference between the resonant phase velocity and the velocity of typical mid-latitude forcing than would result if the wind were the same at all latitudes. If the wind were westerly at all latitudes, the free modes would be Doppler-shifted by approximately the same amount as the forcing motions in mid-latitudes. With shear, however, the mid-latitude motions will be advected by a strong westerly current, in contrast to the free tropical modes that are advected by more easterly winds.

5. FORCED SOLUTIONS WITH PROPAGATION AND ABSORPTION

The behavior of a wave whose phase speed approaches the speed of the zonal current has been studied in other treatments of wave propagation through a shear flow. The fact that the linearized equations exhibit a singularity at that phase speed has led investigators to use terms like "critical layer" (Booker and Bretherton 1967) or "singular line" (Dickinson 1968). Because of the singularity, attempts to treat the problem using numerical methods have led to some confusion in the literature. Hines and Reddy (1967) discussed the alternative physical interpretations of the critical layer effect in terms of reflection processes, while Bretherton (1966) discussed the problem in terms of classical ray tracing theory and found absorption of wave energy. The matter was finally resolved when Booker and Bretherton (1967) showed that an alternative approach, in which a simple model is allowed to achieve a steady state after being started from rest, led to the absorption predicted by Bretherton.

The problem we have is actually both physically and mathematically more like the problem of Dickinson (1968) since it occurs for horizontal shear flow and has the same type of singularity. (The coefficient of v in eq (16) goes to infinity as $U\rightarrow c$ like $1/(U-c)$; in the gravity wave theory, the analogous coefficient goes like $1/(U-c)^2$.) Dickinson noted that the singularity appears to invalidate any planetary wave theory in which the waves are artificially confined between reflecting barriers. He states that there are no normal modes of a "singular wave guide" (i.e., a region bounded by critical layers). Dickinson (1970) has also solved the problem considered here for the long-wave approximation (i.e., the term k^2 of eq 27 is neglected) and obtained results consistent with ours.

The solution of the "asymptotic" problem contains many mathematical difficulties and still fails to answer several fundamental questions. Hines and Reddy (1967) considered an atmospheric model composed of discrete layers and matched solutions at the interfaces. This gave controversial results because it could not adequately allow for the fact that the coefficient actually gets arbitrarily large over a very small distance. Jones (1967) solved the asymptotic problem by adding a small imaginary component (c_i) to the phase speed (corresponding to Rayleigh damping) so that $1/(U-c)$ became bounded in magnitude by $1/|c_i|$. Booker and Bretherton (1967) attempted to solve an initial value problem to generate the asymptotic solution. They used a Laplace transform approach but were only able to evaluate the resulting integral equation for large times, thereby losing vital information about the wave development in the critical layer.

The development in a critical layer or singular line is important for two reasons: (1) it gives information about the time scales needed for development and the manner in which the final state is reached and (2) it tells one how long it takes for the linearization to become invalid. The asymptotic solutions imply an infinite energy density in the vicinity of the critical layer and thus cast doubt upon the validity of the solutions. If one can show that this phenomenon occurs only after unrealistically long times, then these solutions need not be dismissed as a mathematical curiosity.

To facilitate the solution of the mathematical problem, we find it convenient to assume the flow to be nondivergent. This, of course, removes gravity waves from our model, but at the length and time scales we have been working with, they have not really been important, as we saw in the scaling in section 2. The nondivergent assumption is also justified by Charney's (1963) scale analysis of large-scale tropical motions. At any rate, it will be shown that it leads to the same type of singularity as in eq (16) but allows a much simpler initial value problem.

If the flow is horizontally nondivergent, a stream function can be defined such that

$$u'=-\frac{\partial\psi'}{\partial y},\ v'=\frac{\partial\psi'}{\partial x}. \tag{24}$$

Using these relations and eq (1) and (2), we can derive the linearized barotropic vorticity equation valid for nondivergent flow at any latitude:

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2\psi'+\left(\beta-\frac{d^2U}{dy^2}\right)\frac{\partial\psi'}{\partial x}=0. \tag{25}$$

One can see how this leads to an equation like eq (16) by assuming solutions of the form

$$\psi'(x,y,t)=\mathrm{Re}[P(y)e^{i(kx-kct)}].$$

The resulting equation is

$$\frac{d^2P}{dy^2}+\left(-k^2+\frac{1}{U-c}\left(\beta-\frac{d^2U}{dy^2}\right)\right)P=0. \tag{26}$$

Comparison with eq (16) indicates the formal assumption of complete nondivergence has eliminated the $-y^2$ and $y^2U/(U-c)$ terms in the coefficient of v in eq (16). Near a critical layer (where $1/(U-c)$ becomes dominant), the coefficient of v in eq (26) becomes singular in a manner similar to that in eq (16). Thus, the nondivergent model behaves like the divergent one near the critical latitude.

Since we wish to consider an initial value problem, the solutions are periodic in time only in an asymptotic sense. Thus, we assume periodic solutions only in x:

$$\psi'(x,y,t)=\mathrm{Re}\,[\psi(y,t)e^{ikx}].$$

This leads to

$$\left[\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial y^2}-k^2\right)\right]\psi=-Uik\left(\frac{\partial^2}{\partial y^2}-k^2\right)\psi-\beta^*ik\psi \tag{27}$$

where $\beta^*=\beta-(d^2U/dy^2)$.

The method adopted for the solution of this equation is that known as the modified Euler technique discussed by Lilly (1965) and Young (1968). Lilly found the method desirable for ordinary differential equations because it is neutrally stable and conserves mean square vorticity for spectral models. In extending its use to a partial differential equation, it retains computational stability. This difference scheme for the time dependence is

$$\left[\left(\frac{\partial^2}{\partial y^2}-k^2\right)\psi\right]^{n+1}=\left[\left(\frac{\partial^2}{\partial y^2}-k^2\right)\psi\right]^{n}-\frac{ik\Delta t\beta^*}{2}[\psi^{n+1}+\psi^n]$$
$$-\frac{\Delta tUik}{2}\left\langle\left[\left(\frac{\partial^2}{\partial y^2}-k^2\right)\psi\right]^{n+1}+\left[\left(\frac{\partial^2}{\partial y^2}-k^2\right)\psi\right]^{n}\right\rangle \tag{28}$$

where $t=n\Delta t$. Centered differences are used in y:

$$\frac{\partial^2\psi}{\partial y^2}\doteq\frac{\psi_{i+1}+\psi_{i-1}-2\psi_i}{(\Delta y)^2} \tag{29}$$

where $y=i\Delta y$.

Using eq (28) and (29) and rearranging to place all $n+1$ terms on the left-hand side, we obtain

$$\left(1+\frac{\Delta tUik}{2}\right)\left[\frac{\psi_{i+1}^{n+1}+\psi_{i-1}^{n+1}-2\psi_i^{n+1}}{(\Delta y)^2}-k^2\psi_i^{n+1}\right]+\frac{\Delta t\beta^*ik}{2}\psi_i^{n+1}$$
$$=\left(1-\frac{\Delta tUik}{2}\right)\left[\frac{\psi_{i+1}^{n}+\psi_{i-1}^{n}-2\psi_i^{n}}{(\Delta y)^2}-k^2\psi_i^{n}\right]-\frac{\Delta t\beta^*ik}{2}\psi_i^{n}. \tag{30}$$

It is assumed that ψ is initially zero everywhere from $-D\leq y<+D$ but that a disturbance $\psi(D,t)=1$ is raised along the northern boundary at $t=0$ and thereafter kept constant with time. Alternatively, one could apply a periodic forcing in time to simulate a wave traveling along the boundary in the manner of Houghton and Jones (1969). However, since eq (25) is invariant under a Galilean transformation of the type

$$U(y)\rightarrow U(y)+U'$$

where U' is constant, the two approaches are equivalent. The initial and boundary conditions are, therefore,

$$\psi(y, 0)=0 \qquad \text{for } -D\leq y<+D,$$

$$\psi(D, 0)=1.0,$$

$$\psi(D, t)=1.0,$$

and

$$\psi(-D, t)=0.0.$$

The last equation implies a reflecting wall at $y=-D$ since $v=0$ there; its effect is negligible here since very little energy propagates to that latitude in this solution.

With the boundary conditions, there is a complete set of linear algebraic equations, one for each internal grid point, which determine ψ from its values for the preceding time step. The system can be written as

$$\mathbf{A}\psi^{n+1}=\mathbf{b}^n.$$

If m is the number of internal grid points, $\mathbf{A}$ is an $m\times m$ matrix of coefficients that is constant with time; ψ^{n+1} is a column vector of m elements; and $\mathbf{b}^n$ is another column vector of m elements that depends on the values of ψ_i^n. Since ψ is complex, this is equivalent to the solution of $2\,m$ algebraic equations for real variables. In this work, 49 internal grid points were used, resulting in 98 algebraic equations. The fact that $\mathbf{A}$ does not vary with time means that one may perform a Gaussian elimination on it only once and simply back substitute at each time step when there is a new term $\mathbf{b}^n$.

As an aid to the interpretation of the solutions to this problem, it is helpful to consider what classical wave theory tells us about the problem. It is well known that eq (25) allows only stable Rossby waves unless $[\beta-(d^2U/dy^2)]$ is zero somewhere. Using WKB logic, we may consider U to be locally constant (see Morse and Feshbach 1953). Solutions of the form

$$\psi=\mathrm{Re}\,[\hat{\psi}e^{i(kx+ly-\omega t)}]$$

satisfy the dispersion relation:

$$\omega=Uk-\frac{(\beta-U'')k}{k^2+l^2}. \tag{31}$$

By setting $\psi=1.0$ at the boundary, we have specified its phase to be constant in time. Thus the frequency should be zero there, and eq (31) determines a value for l^2:

$$l^2=-k^2+\frac{\beta-U''}{U(y)}. \tag{32}$$

In this manner, the two conditions at the boundary (that the wave be stationary with a given zonal wind) determine a wavelength in y. From the dispersion relation (31), one can compute a group velocity for the y direction:

$$C_{gy}=\frac{\partial\omega}{\partial l}=\frac{2kl(\beta-U'')}{(k^2+l^2)^2}. \tag{33}$$

Thus the zonal wind at the boundary indirectly predicts an initial group velocity. With only qualitative accuracy, we can assume that the disturbance excited at the northern boundary will propagate toward the Equator with a local group velocity given by eq (33) and a structure $l^2\,(y)$ given by eq (32).

Since we are mainly concerned with the behavior near a critical layer, it will suffice to consider a linear shear flow. For convenience, we will assume it to have a zero at $y=0$. Thus

$$U(y)=y\left(\frac{dU}{dy}\right) \tag{34}$$

where $dU/dy=$constant. The latitude $y=0$ therefore is the critical latitude for the wave with speed $c=0$.

Noticing that l becomes large near $y=0$, we can approximate eq (32) and (33) thereby:

$$l^2\doteq\frac{\beta-U''}{y\,\dfrac{dU}{dy}}$$

and

$$C_{gy}\doteq(\beta-U'')^{-1/2}\left(\frac{dU}{dy}\right)^{3/2}2ky^{3/2}. \tag{35}$$

If a wave group approaches $y=0$ from $y=D$ at this velocity, the time it takes to get there will be

$$t\doteq\frac{(\beta-U'')^{1/2}\left(\dfrac{dU}{dy}\right)^{-3/2}}{2k}\int_0^D y^{-3/2}dy. \tag{36}$$

The fact that this integral is not finite leads to the concept of absorption since the group is neither reflected nor transmitted; it is simply retarded. This explanation gives the right behavior for the phenomenon, but the WKB theory is wrong in principle. For a WKB approximation to be valid, the local wave number l should not vary much over a wavelength ($=2\pi/l$). In other words, the percentage change in wave number over a wavelength L_y, given by

$$\frac{\dfrac{\partial l}{\partial y}L_y}{l}=\frac{\partial l}{\partial y}\frac{2\pi}{l^2},$$

must be of order unity or less. In our case, (35) shows that l is proportional to $y^{-1/2}$ near the critical level. Thus near $y=0$, the above expression gives

$$\frac{2\pi}{l^2}\frac{\partial l}{\partial y}=\frac{2\pi\dfrac{dU}{dy}y}{\beta-U''}\frac{\partial}{\partial y}\left(\frac{(\beta-U'')^{1/2}}{y^{1/2}}\right)$$

$$=-\pi\frac{dU}{dy}(\beta-U'')^{-1/2}y^{-1/2} \tag{37}$$

which goes to infinity as y goes to zero; thus the WKB

TABLE 2.—*Physical parameters for the three cases studied numerically*

Case	k	$l(D)$	$U(D)$	dU/dy	C_{gy}
A	2.1	6.5	0.0225	0.015	0.0133
B	2.1	4.0	.05	.033	.042
C	.7	1.7	.3	.2	.20

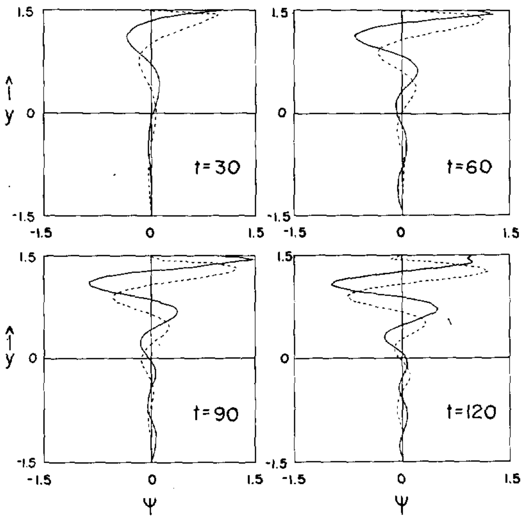

FIGURE 9.—Real (solid lines) and imaginary (dashed lines) parts of the stream function $\psi(y)$ of case A for initial stages of development. The variables specified are k, an east-west wave number, and dU/dy, the shear of the mean wind.

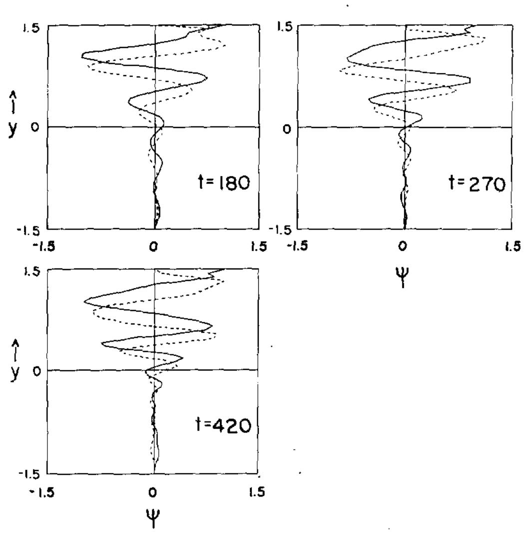

FIGURE 10.—Same as figure 9, except for later times.

approximation is invalidated near the singularity. However, it can still give us a first estimate of the rate of energy propagation away from the singularity.

Since we have chosen a linear wind profile and have chosen it to be zero at $y=0$, the parameters $l(D)$, $U(D)$, dU/dy, and $C_{gy}(D)$ are all related. If any one of them is known along with the east-west wave number k, the other three can be computed. For convenience, however, they have all been calculated in table 2 for the three numerical experiments we shall consider.

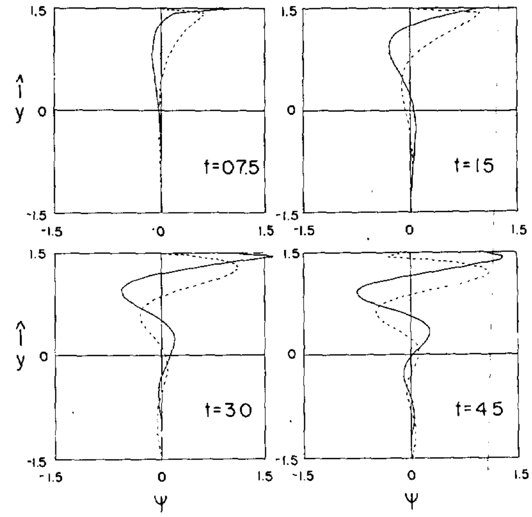

FIGURE 11.—Real (solid lines) and imaginary (dashed lines) parts of the stream function $\psi(y)$ of case B for initial stages of development. The variables specified are k, an east-west wave number, and dU/dy, the shear of the mean wind.

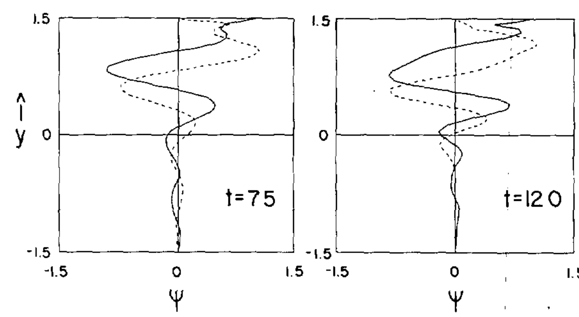

FIGURE 12.—Same as figure 11, except for later times.

Cases A and B differ only in the mean shear dU/dy (and hence $U(D)$). By our WKB logic, we expect ψ in case B to have a larger wavelength in y and hence to propagate faster toward the Equator. This is readily verified by inspection of figures 9 through 14. However, although the WKB method also predicts that the wavelength near the critical level becomes extremely small, it does not occur in the numerical solutions. In the quasi-steady solutions (the last figure of each series), no drastic shortening of wavelength near $y=0$ is apparent.

One can scale this problem as the problems in sections 3 and 4 by choosing $\sqrt{g'H_0}$ to be 100 m s^{-1}. This gives a length scale, $\hat{L}$, from eq (6) of approximately 2100 km and a corresponding time scale factor $\hat{T}$ of 5.85 hr. At this scaling, cases A and B correspond to waves in which east-west wavelength is approximately 6300 km, and C corresponds to a wavelength of 19,000 km.

In the initial stages of development, there is a rapid growth in the magnitude of ψ throughout the whole domain as a response to the impulsive start. This response dies off exponentially away from the forced boundary in a manner similar to potential flow. The initially irrotational motion satisfies

$$\left(\frac{\partial^2}{\partial y^2}-k^2\right)\psi=0.$$

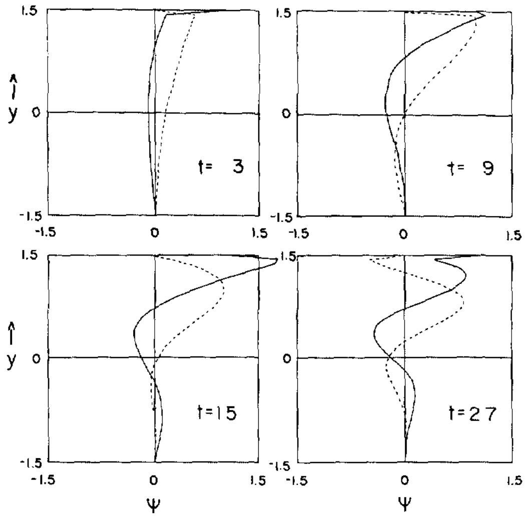

FIGURE 13.—Real (solid lines) and imaginary (dashed lines) parts of the stream function $\psi(y)$ of case C for initial stages of development. The variables specified are k, an east-west wave number, and dU/dy, the shear of the mean wind.

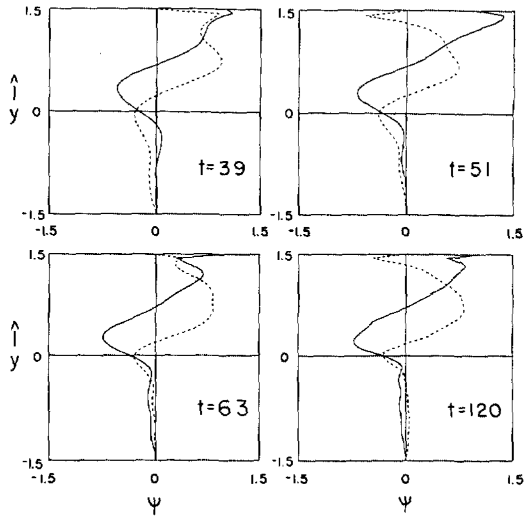

FIGURE 14.—Same as figure 13, except for later times.

Solutions of this equation are

$$\psi=e^{+ky},\ e^{-ky}.$$

This relation can be verified quantitatively for case C at $t=3.0$. After this initial effect, the solution becomes oscillatory and the disturbance begins to propagate southward. This propagation slows down as the disturbance approaches $y=0$, where the singularity exists in the asymptotic problem. Eventually, the solution reaches a quasi-steady state in which it has negligible amplitude on the southern side of the critical latitude. This solution remains essentially unchanged even for times three to four times that needed for development.

The essential difference between the solutions for cases A and B is that they show different characteristic wavelengths in y. The fact that there is a shorter wavelength in case A verifies our prediction that this wavelength should be consistent with the classical Rossby wave formula for stationary disturbances:

$$U=\frac{\beta}{k^2+l^2}.$$

The three cases are consistent with the prediction that the time scales for development to steady states should depend on the group velocity at the boundary. When using long-period calculations, it was found that case A attained a quasi-steady state at a time of 420 nondimensional units; case B took 120 units; and case C, about 60.

We can assess the validity of the linearization by asking whether the value of $\partial\psi/\partial y$, which corresponds to u', becomes much larger than u' at the boundary. In case C, there is a tendency for the derivative of ψ to get large, but it never exceeds twice the boundary value even when the calculations were carried out to large times compared to the time needed for steady development.

An important property of the solutions is the variation of ψ phase with latitude, which corresponds to a tilt of the troughs and ridges from southwest to northeast. This tilt is indicative of a northward transport of zonal momentum. Application of the theory of Eliassen and Palm (1961) to this model shows that this northward flux of momentum implies a southward energy flux, suggesting that reflection is not too important. The relatively small values of ψ for $y<0$ indicate that little energy is transmitted beyond the critical latitude. Thus we conclude that the wave-mean flow interaction represents an absorption of wave energy.

6. SUMMARY AND CONCLUDING REMARKS

The atmosphere was idealized to isolate the effect of the horizontal shear of the wind on the interaction between mid-latitude regions and the Tropics as a single layer of hydrostatic fluid with a free surface. The problem without basic zonal wind was first solved to make a numerical comparison between motions with and without shear for the free modes of the Tropics. A resonance phenomenon was examined by forcing the boundary of the model with a wave disturbance of a specified phase velocity. As expected, the largest response in the Tropics occurred when the forcing velocity was close to the phase velocity of a free mode (i.e., to the west).

It was not convenient to solve for the free modes in a direct manner when the effects of a zonal wind shear were included. Instead, they were inferred from the resonant response to the forced problem. The forced problem was treated only for those phase velocities for which the governing equation was not singular. Because the equation was singular for all phase velocities which were neither large eastward nor large westward, it was left for a special analysis. The resonant responses to the nonsingular forced problem indicated that the main effect of the zonal wind was to Doppler shift the phase speeds of the free modes in the direction of the wind averaged over all latitudes of the model.

Because of this Doppler shift, the observed winds appeared to inhibit interaction between the Tropics and mid-latitudes. If the westerlies of mid-latitudes extended to the Equator, the free modes of the Tropics would move with phase speeds closer to those of mid-latitude disturbances and hence be closer to resonance.

The important case that could not be treated by the above analysis occurred when the longitudinal phase velocity of a propagating wave became equal to the mean zonal flow at some "critical" latitude. This case was treated by simplifying the model slightly by assuming the flow to be nondivergent. This assumption made it feasible to pose a problem in which the northern boundary of the initially undisturbed Tropics was forced and the motion computed as a function of time and latitude. The numerical solutions to this marching problem indicated that singular line absorption of wave energy can occur at critical latitudes. Time scales for this energy loss from mid-latitude motions range from 2–3 days for planetary scales to several weeks for small scales (1000 km).

The observed shear of the zonal wind implies that this mechanism serves to inhibit the propagation of wave disturbances, just as the Doppler shift of the free modes does. An important difference, however, is that the Doppler shift mechanism does not affect the mean zonal wind, in contrast to the strong influence of the eddy motions implied in the absorption mechanism. Interestingly, there are both observational and theoretical grounds that indicate the importance of the absorption mechanism.

The observed eddy motions transport eastward momentum into the latitudes where the zonal winds are a maximum (Mak 1969). The observed energy flux (energy flux here is defined as $\overline{p'v'}$, the pressure work done on the fluid north of a latitude circle by the fluid south of it) is southward (Mak 1969) and thus consistent with the momentum flux for the model treated here once a steady state has developed. Thus, the propagation of wave disturbances from mid-latitudes into the Tropics is consistent with the observed momentum and energy fluxes.

Theoretical models have exhibited this observed northward momentum flux and southward energy flux. Mak's (1969) stochastic model also shows a very strong conversion from eddy kinetic energy to zonal kinetic energy in the Tropics, implying that the motions serve to maintain the zonal winds. However, Mak did not discuss in detail the mechanism by which this momentum is taken from the easterly flow. Although he used a two-layer model, his results indicate little vertical coupling; it is thus reasonable to assume that his results can be interpreted in terms of the one-layer model presented here.

Mak found that the eddy motions were much smaller in his model Tropics than in mid-latitudes. The dominant motions had phase speeds directed westward relative to the mean flow. From the present analysis of the forced modes in the presence of shear, these motions are precisely the ones one would expect to propagate, since the resonant responses corresponding to the free modes were always westward-moving. However, the forcing at the boundary of Mak's model included eastward-moving waves. For these waves, the present model predicts singular line absorption and conversion of wave energy to mean flow energy. Interestingly, Mak's model shows very little wave energy in the Tropics for these phase velocities but shows a very strong conversion from eddy kinetic energy to zonal kinetic energy.

In conclusion, the shear of the zonal wind can act to limit the propagation of wave disturbances into the Tropics to those waves traveling more westward than the mean flow at any latitude. Other disturbances, which include most of the eastward-moving or slowly westward-moving distrubances of mid-latitudes, can influence the Tropics by being absorbed by the mean flow and maintaining the easterlies.

ACKNOWLEDGMENTS

The writers profited from discussions with Prof. David D. Houghton of The University of Wisconsin and Dr. Robert E. Dickinson of the National Center for Atmospheric Research.

Financial support was provided by the Environmental Science Services Administration under Grant E–230–68–(G) at the Space Science and Engineering Center of The University of Wisconsin and, for one of us (Bennett), by the National Science Foundation with the University of Colorado. Computing funds were provided by the Wisconsin Alumni Research Foundation at The University of Wisconsin Computing Center and the National Science Foundation at the National Center for Atmospheric Research.

REFERENCES

Abramowitz, M., and Segun, I. A., *Handbook of Mathematical Functions*, Dover Publications, New York, N.Y., **1965**, 1046 pp.

Booker, John R., and Bretherton, Francis, P., "The Critical Layer for Internal Gravity Waves in a Shear Flow," *Journal of Fluid Mechanics*, Vol. 27, Pt. 3, Cambridge University Press, London, England, Feb. 24, **1967**, pp. 513–539.

Bretherton, Francis P., "The Propagation of Groups of Internal Gravity Waves in a Shear Flow," *Quarterly Journal of the Royal Meteorological Society*, Vol. 92, No. 394, London, England, Oct. **1966**, pp. 466–480.

Charney, Jule G., "A Note on Large-Scale Motions in the Tropics," *Journal of the Atmospheric Sciences*, Vol. 20, No. 6, Nov. **1963**, pp. 607–609.

Charney, Jule G., "A further Note on Large-Scale Motions in the Tropics," *Journal of the Atmospheric Sciences*, Vol. 26, No. 1, Jan. **1969**, pp. 182–185.

Charney, Jule G., and Eliassen, Arnt, "On the Growth of the Hurricane Depression," *Journal of the Atmospheric Sciences*, Vol. 21, No. 1, Jan. **1964**, pp. 68–75.

Dickinson, Robert E., "Planetary Rossby Waves Propagating Vertically Through Weak Westerly Wind Wave Guides," *Journal of the Atmospheric Sciences*, Vol. 25, No. 6, Nov. **1968**, pp. 984–1002.

Dickinson, Robert E., "Development of a Rossby Wave Critical Level," *Journal of the Atmospheric Sciences*, Vol. 27, No. 4, July **1970**, pp. 627–633.

Eliassen, Arnt and Palm, Enok, "On the Transfer of Energy in Stationary Mountain Waves," *Geofysiske Publikasjoner*, Vol. 22, No. 3, Oslo, Norway, Sept. **1961**, 23 pp.

Hines, C. O., and Reddy, C. A., "On the Propagation of Atmospheric Gravity Waves Through Regions of Wind Shear," *Journal of Geophysical Research*, Vol. 72, No. 3, Feb. 1, **1967**, pp. 1015–1034.

Houghton, David D., and Jones, Walter L., "A Numerical Model for Linearized Gravity and Acoustic Waves," *Journal of Computational Physics*, No. 3, Academic Press, Inc., New York, N.Y., Feb. **1969**, pp. 339–357.

Jones, Walter L., "Propagation of Internal Gravity Waves in Fluids With Shear Flow and Rotation," *Journal of Fluid Mechanics*, Vol. 30, Pt. 3, Cambridge University Press, London, England, Nov. 29, **1967**, pp. 439–448.

Koss, Walter J., "Further Theoretical Considerations of Tropospheric Wave Motions in Equatorial Latitudes," *Monthly Weather Review*, Vol. 95, No. 5, May **1967**, pp. 283–297.

Lilly, Douglas K., "On the Computational Stability of Numerical Solutions of Time-Dependent Non-Linear Geophysical Fluid Dynamics Problems," *Monthly Weather Review*, Vol. 93, No. 1, Jan. **1965**, pp. 11–26.

Lindzen, Richard S., "Planetary Waves on Beta-Planes," *Monthly Weather Review*, Vol. 95, No. 7, July **1967**, pp. 441–451.

Lipps, Frank B., "Barotropic Stability and Tropical Disturbances *Monthly Weather Review*, Vol. 98, No. 2, Feb. **1970**, pp. 122–131.

Mak, Man Kin, "Laterally Driven Stochastic Motions in the Tropics," *Journal of the Atmospheric Sciences*, Vol. 26, No. 1, Jan. **1969**, pp. 41–64.

Maruyama, Taketo, "Time Sequence of Power Spectra Disturbances in the Equatorial Lower Stratosphere in Relation to the Quasi-Biennial Oscillation," *Journal of the Meteorological Society of Japan*, Vol. 46, No. 5, Tokyo, Oct. **1968**, pp. 404–417.

Matsuno, Taroh, "Quasi-Geostrophic Motions in the Equatorial Area," *Journal of the Meteorological Society of Japan*, Ser. 2, Vol. 44, No. 1, Tokyo, Feb. **1966**, pp. 25–42.

Miles, J. W., "On the Generation of Surface Waves by Shear Flows, Part 4," *Journal of Fluid Mechanics*, Vol. 13, No. 3, July **1962**, pp. 433–448.

Morse, P.M., and Feshbach, H., *Methods of Theoretical Physics*, McGraw-Hill Book Co., Inc., New York, N.Y., **1953**, 1978 pp. (see p. 1092).

Nitta, Tsuyoshi, and Yanai, M., "A Note on the Barotropic Instability of the Tropical Easterly Current," *Journal of the Meteorological Society of Japan*, Vol. 47, No. 2, Tokyo, Apr. **1969**, pp. 127–130.

Rosenthal, Stanley L., "Some Estimates of the Power Spectra of Large-Scale Disturbances in Low Latitudes," *Journal of Meteorlogy*, Vol. 17, No. 3, June **1960**, pp. 259–263.

Rosenthal, Stanley L., "Some Preliminary Theoretical Considerations of Tropospheric Wave Motions in Equatorial Latitudes," *Monthly Weather Review*, Vol. 93, No. 10, Oct. **1965**, pp. 605–612.

Rosenthal, Stanley L., "On the Development of Synoptic-Scale Disturbances Over the Subtropical Oceans," *Monthly Weather Review*, Vol. 95, No. 6, June **1967**, pp. 341–346.

Wallace, John M., "Some Recent Developments in the Study of Tropical Wave Disturbances," *Bulletin of the American Meteorological Society*, Vol. 50, No. 10, Oct. **1969**, pp. 792–799.

Young, John A., "Comparative Properties of Some Time Differencing Schemes for Linear and Nonlinear Oscillations," *Monthly Weather Review*, Vol. 96, No. 6, June **1968**, pp. 357–364.

[Received March 13, 1970; revised June 24, 1970]